

## 2023 年全国硕士研究生招生考试（数学三）试题及答案解析

1. 已知函数  $f(x, y) = \ln(y + |x \sin y|)$ , 则

- A.  $\left. \frac{\partial f}{\partial x} \right|_{(0,1)}$  不存在,  $\left. \frac{\partial f}{\partial y} \right|_{(0,1)}$  存在.      B.  $\left. \frac{\partial f}{\partial x} \right|_{(0,1)}$  存在,  $\left. \frac{\partial f}{\partial y} \right|_{(0,1)}$  不存在.
- C.  $\left. \frac{\partial f}{\partial x} \right|_{(0,1)}$ ,  $\left. \frac{\partial f}{\partial y} \right|_{(0,1)}$  均存在.      D.  $\left. \frac{\partial f}{\partial x} \right|_{(0,1)}$ ,  $\left. \frac{\partial f}{\partial y} \right|_{(0,1)}$  均不存在.

**【答案】 A**

**【解析】**

$$f(x, y) = \ln(y + |x \sin y|), \quad f(0, 1) = \ln(1 + 0) = 0, \quad f(x, 1) = \ln(1 + |x \sin 1|)$$

$$f(0, y) = \ln(1 + 0) = 0$$

$$\left. \frac{\partial f}{\partial x} \right|_{(0,1)} = \lim_{x \rightarrow 0} \frac{f(x, 1) - f(0, 1)}{x - 0} = \lim_{x \rightarrow 0} \frac{\ln(1 + x \sin 1)}{x} = \lim_{x \rightarrow 0} \frac{|x \sin 1|}{x} \text{ 不存在}$$

$$\left. \frac{\partial f}{\partial y} \right|_{(0,1)} = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 1)}{y - 0} = 0$$

故选 A

2. 函数  $f(x) = \begin{cases} \frac{1}{\sqrt{1+x^2}}, & x \leq 0, \\ (x+1)\cos x, & x > 0 \end{cases}$  的一个原函数为

$$\text{A. } F(x) = \begin{cases} \ln(\sqrt{1+x^2} - x), & x \leq 0, \\ (x+1)\cos x - \sin x, & x > 0. \end{cases}$$

$$\text{B. } F(x) = \begin{cases} \ln(\sqrt{1+x^2} - x) + 1, & x \leq 0, \\ (x+1)\cos x - \sin x, & x > 0. \end{cases}$$

$$\text{C. } F(x) = \begin{cases} \ln(\sqrt{1+x^2} + x), & x \leq 0, \\ (x+1)\sin x + \cos x, & x > 0. \end{cases}$$

$$\text{D. } F(x) = \begin{cases} \ln(\sqrt{1+x^2} + x) + 1, & x \leq 0, \\ (x+1)\sin x + \cos x, & x > 0. \end{cases}$$

**【答案】** D

**【解析】**

$$\int (x+1) \cos x dx = \int (x+1) d \sin x = (x+1) \sin x - \int \sin x dx = (x+1) \sin x + \cos x + c$$

故排除 AB, 由于  $\lim_{x \rightarrow 0^+} F(x) = 1 \neq \lim_{x \rightarrow 0^-} F(x) = 0$ , 排除 C, 故选 D

3. 若微分方程  $y'' + ay' + by = 0$  的解在  $(-\infty, +\infty)$  上有界, 则

A.  $a < 0, b > 0$ .

B.  $a > 0, b > 0$ .

C.  $a = 0, b > 0$ .

D.  $a = 0, b < 0$ .

**【答案】** C

**【解析】** 当  $y'' + ay' + by = 0$  有实根时,  $a^2 - 4b \geq 0$ , 设根为  $r_1, r_2$ , 则  $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$  或

$y = (c_1 + c_2 r) e^{r_1 x} (r_1 = r_2)$ . 故此时不可能有解在  $(-\infty, +\infty)$  有界. 当  $a^2 - 4b < 0$  时.

$y = (c_1 \cos \beta x + c_2 \sin \beta x) e^{ax}$ , 若想解在  $(-\infty, +\infty)$  有界, 因此  $a = 0$ , 结合  $a^2 - 4b < 0$  可得

$b > 0$ . 故选 C.

4. 已知  $a_n < b_n (n = 1, 2, \dots)$ , 若级数  $\sum_{n=1}^{\infty} a_n$  与  $\sum_{n=1}^{\infty} b_n$  均收敛, 则“ $\sum_{n=1}^{\infty} a_n$  绝对收敛”是“ $\sum_{n=1}^{\infty} b_n$  绝对收敛”的

对收敛”的

A. 充分必要条件.

B. 充分不必要条件.

C. 必要不充分条件.

D. 既不充分也不必要条件.

**【答案】** A

**【解析】** 由级数  $\sum_{n=1}^{\infty} a_n$  与  $\sum_{n=1}^{\infty} b_n$  均收敛, 可知  $\sum_{n=1}^{\infty} |a_n - b_n|$  收敛.

若  $\sum_{n=1}^{\infty} b_n$  绝对收敛, 由  $|a_n| = |b_n + a_n - b_n| \leq |b_n| + |a_n - b_n|$ , 可知  $\sum_{n=1}^{\infty} a_n$  绝对收敛.

若  $\sum_{n=1}^{\infty} a_n$  绝对收敛,  $|b_n| = |a_n + b_n - a_n| \leq |a_n| + |b_n - a_n|$ , 可知  $\sum_{n=1}^{\infty} b_n$  绝对收敛.

故选 A

5. 设  $A, B$  为  $n$  阶可逆矩阵,  $E$  为  $n$  阶单位矩阵,  $M^*$  为矩阵  $M$  的伴随矩阵, 则  $\begin{pmatrix} A & E \\ O & B \end{pmatrix}^* =$

A.  $\begin{pmatrix} |A|B^* & -B^*A^* \\ O & |B|A^* \end{pmatrix}$ .

B.  $\begin{pmatrix} |B|A^* & -A^*B^* \\ O & |A|B^* \end{pmatrix}$ .

C.  $\begin{pmatrix} |B|A^* & -B^*A^* \\ O & |A|B^* \end{pmatrix}$ .

D.  $\begin{pmatrix} |A|B^* & -A^*B^* \\ O & |B|A^* \end{pmatrix}$ .

**【答案】** D

**【解析】**  $\begin{bmatrix} A & E \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} A & E \\ 0 & B \end{bmatrix}^{-1} \cdot \begin{bmatrix} A & E \\ 0 & B \end{bmatrix}^{-1}$

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} A & E \\ 0 & B \end{bmatrix} = \begin{bmatrix} X_1A & X_1 + X_2B \\ X_3A & X_3 + X_4B \end{bmatrix}$$

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \begin{bmatrix} A^{-1} & -A^{-1}B^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A & E \\ 0 & B \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}B^{-1} \\ 0 & B^{-1} \end{bmatrix}$$

$$\begin{bmatrix} A & E \\ 0 & B \end{bmatrix}^* = |A| \cdot |B| \cdot \begin{bmatrix} A^{-1} & -A^{-1}B^{-1} \\ 0 & B^{-1} \end{bmatrix} = \begin{bmatrix} |B| \cdot A^* & -A^*B^* \\ 0 & |A| \cdot B^* \end{bmatrix}$$

6 二次型  $f(x_1, x_2, x_3) = (x_1 + x_2)^2 + (x_1 + x_3)^2 - 4(x_2 - x_3)^2$  的规范形为

A.  $y_1^2 + y_2^2$

B.  $y_1^2 - y_2^2$

C.  $y_1^2 + y_2^2 - 4y_3^2$

D.  $y_1^2 + y_2^2 - y_3^2$

**答案:** B.

**【解析】**  $f(x_1, x_2, x_3) = 2x_1^2 - 3x_2^2 - 3x_3^2 + 2x_1x_2 + 2x_1x_3 + 8x_2x_3$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -3 & 4 \\ 1 & 4 & -3 \end{bmatrix}$$

$$|A - \lambda E| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & -3-\lambda & 4 \\ 1 & 4 & -3-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & -3-\lambda & 4 \\ 0 & 7+\lambda & -7-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 1 & -3-\lambda & 1-\lambda \\ 0 & 7+\lambda & 0 \end{vmatrix}$$

$= (7 + \lambda)\lambda(3 - \lambda)$ . 故选 B

7. 已知向量  $\alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ,  $\alpha_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ ,  $\beta_1 = \begin{pmatrix} 2 \\ 5 \\ 9 \end{pmatrix}$ ,  $\beta_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ , 若  $\gamma$  既可由  $\alpha_1, \alpha_2$  线性表示, 也可由

$\beta_1, \beta_2$  线性表示, 则  $\gamma =$

- A.  $k \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, k \in \mathbf{R}$       B.  $k \begin{pmatrix} 3 \\ 5 \\ 10 \end{pmatrix}, k \in \mathbf{R}$       C.  $k \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, k \in \mathbf{R}$       D.  $k \begin{pmatrix} 1 \\ 5 \\ 8 \end{pmatrix}, k \in \mathbf{R}$

**【答案】D**

**【解析】**

$$\gamma = k_1\alpha_1 + k_2\alpha_2 = l_1\beta_1 + l_2\beta_2, \quad k_1\alpha_1 + k_2\alpha_2 - l_1\beta_1 - l_2\beta_2 = 0,$$

$$\begin{cases} x_1 = k_1 \\ x_2 = k_2 \\ x_3 = -l_1 \\ x_4 = -l_2 \end{cases} \quad x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + x_4\alpha_4 = 0$$

$$\begin{cases} x_1 = 3k \\ x_2 = -k \end{cases}, \gamma = 3k \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - k \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = k \begin{bmatrix} 1 \\ 5 \\ 8 \end{bmatrix}$$

8. 设随机变量  $X$  服从参数为 1 的泊松分布, 则  $E(|X - EX|) =$

- A.  $\frac{1}{e}$       B.  $\frac{1}{2}$       C.  $\frac{2}{e}$       D. 1

**【答案】C**

**【解析】**  $E(|X - 1|)$

$$= E(X - 1) + 2 \cdot P\{X = 0\}$$

$$= 0 + 2e^{-1} = 2e^{-1}$$

9. 设  $X_1, X_2, \dots, X_n$  为来自总体  $N(\mu_1, \sigma^2)$  的简单随机样本,  $Y_1, Y_2, \dots, Y_m$  为来自总体

$N(\mu_2, 2\sigma^2)$  的简单随机样本, 且两样本相互独立, 记

$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i, \bar{Y} = \frac{1}{m} \sum_{i=1}^m Y_i, S_1^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2, S_2^2 = \frac{1}{m-1} \sum_{i=1}^m (Y_i - \bar{Y})^2$ , 则

A.  $\frac{S_1^2}{S_2^2} \sim F(n, m)$

B.  $\frac{S_1^2}{S_2^2} \sim F(n-1, m-1)$

C.  $\frac{2S_1^2}{S_2^2} \sim F(n, m)$

D.  $\frac{2S_1^2}{S_2^2} \sim F(n-1, m-1)$

【答案】D

$$\frac{(n-1)S_1^2}{\sigma^2} \sim \chi^2(n-1) \quad \frac{(m-1)S_2^2}{2\sigma^2} \sim \chi^2(m-1)$$

【解析】  $\frac{\frac{(n-1)S_1^2}{\sigma^2} / n-1}{\frac{(m-1)S_2^2}{2\sigma^2} / m-1} = \frac{2S_1^2}{S_2^2} \sim F(n-1, m-1)$

10. 设  $X_1, X_2$  为来自总体  $N(\mu, \sigma^2)$  的简单随机样本, 其中  $\sigma (\sigma > 0)$  是未知参数. 记

$\sigma = a|X_1 - X_2|$ , 若  $E(\sigma) = \sigma$ , 则  $a =$

A.  $\frac{\sqrt{\pi}}{2}$

B.  $\frac{\sqrt{2\pi}}{2}$

C.  $\sqrt{\pi}$

D.  $\sqrt{2\pi}$

10 【答案】A

【解析】  $Ea|X_1 - X_2| = aE|X_1 - X_2| = a \cdot \frac{2\sigma}{\sqrt{\pi}} = \sigma \quad a = \frac{\sqrt{\pi}}{2}$

其中:  $X_1 - X_2 \sim N(0, 2\sigma^2)$ , 令  $Z = X_1 - X_2$

$$\begin{aligned} E|X_1 - X_2| &= \int_{-\infty}^{+\infty} |z| \cdot \frac{1}{\sqrt{2\pi}\sqrt{2\sigma}} \cdot e^{-\frac{z^2}{4\sigma^2}} dz = 2 \int_0^{+\infty} \frac{z}{2\sqrt{\pi}\sigma} e^{-\frac{z^2}{4\sigma^2}} dz \\ &= 2 \frac{1}{2\sqrt{\pi}\sigma} (-2\sigma^2) \int_0^{+\infty} e^{-\frac{z^2}{4\sigma^2}} d\left(-\frac{z^2}{4\sigma^2}\right) = -\frac{2\sigma}{\sqrt{\pi}} e^{-\frac{z^2}{4\sigma^2}} \Bigg|_0^{+\infty} = \frac{2\sigma}{\sqrt{\pi}} \end{aligned}$$

二、填空题

$$11. \lim_{x \rightarrow \infty} x^2 \left( 2 - x \sin \frac{1}{x} - \cos \frac{1}{x} \right) = \underline{\hspace{2cm}}.$$

**【答案】**  $\frac{2}{3}$

**【解析】**

$$\begin{aligned} \lim_{x \rightarrow \infty} x^2 \left( 2 - x \sin \frac{1}{x} - \cos \frac{1}{x} \right) &= \lim_{t \rightarrow 0} \frac{1}{t^2} \left( 2 - \frac{1}{t} \sin t - \cos t \right) = \lim_{t \rightarrow 0} \frac{2t - \sin t - t \cos t}{t^3} \\ &= \lim_{t \rightarrow 0} \frac{2t - \left( t - \frac{1}{6}t^3 \right) - t \left( 1 - \frac{1}{2}t^2 \right) + o(t^3)}{t^3} = \lim_{t \rightarrow 0} \frac{\frac{1}{6}t^3 + \frac{1}{2}t^3}{t^3} = \frac{2}{3} \end{aligned}$$

$$12. \text{已知函数 } f(x, y) \text{ 满足 } df(x, y) = \frac{xdy - ydx}{x^2 + y^2}, f(1, 1) = \frac{\pi}{4}, \text{ 则 } f(\sqrt{3}, 3) = \underline{\hspace{2cm}}.$$

**【答案】**  $\frac{\pi}{12}$

**【解析】**  $\frac{\partial f}{\partial x} = \frac{-y}{x^2 + y^2}, f(x, y) = -\arctan \frac{x}{y} + c(y)$

$$\frac{\partial f}{\partial y} = -\frac{x}{1 + \frac{x^2}{y^2}} + c'(y) = \frac{x}{x^2 + y^2} + c'(y), \frac{\partial f}{\partial y} = \frac{x}{x^2 + y^2}, \text{ 故 } c'(y) = 0, c(y) = 0$$

$$f(1, 1) = \frac{\pi}{4}, c = \frac{\pi}{4}, f(x, y) = -\arctan \frac{x}{y} + \frac{\pi}{4}$$

$$f(\sqrt{3}, 3) = -\arctan \frac{\sqrt{3}}{3} + \frac{\pi}{4} = \frac{\pi}{6} - \frac{\pi}{6} = \frac{\pi}{12}$$

$$13. \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \underline{\hspace{2cm}}.$$

**【答案】**  $\frac{e^x + e^{-x}}{2}$

【解析】  $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ,  $e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{n!}$ ,  $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = \frac{e^x + e^{-x}}{2}$ .

14. 设某公司在  $t$  时刻的资产为  $f(t)$ , 从 0 时刻到  $t$  时刻的平均资产等于  $\frac{f(t)}{t} - t$ , 假设  $f(t)$  连续且  $f(0)=0$ , 则  $f(t)=$  \_\_\_\_\_.

【答案】  $f(t) = 2(1-t) - 2e^t$

【解析】

$$f'(t) - 2t = f(t)$$

$$f'(t) = f(t) + 2t$$

$$f'(t) - f(t) = 2t$$

$$f(t) = e^{-\int -1 dt} \left[ \int 2te^{\int -1 dt} dt + c \right] = e^t \left[ \int 2t \cdot e^{-t} dt + c \right]$$

$$= 2e^t \left[ (1-t)e^{-t} + c \right] = 2(1-t) + 2ce^t$$

$$f(0) = 2 + 2c = 0 \Rightarrow c = -1$$

$$f(t) = 2(1-t) - 2e^t$$

15. 已知线性方程组 
$$\begin{cases} ax_1 + x_3 = 1, \\ x_1 + ax_2 + x_3 = 0, \\ x_1 + 2x_2 + ax_3 = 0, \\ ax_1 + bx_2 = 2 \end{cases}$$
 有解, 其中  $a, b$  为常数, 若 
$$\begin{vmatrix} a & 0 & 1 \\ 1 & a & 1 \\ 1 & 2 & a \end{vmatrix} = 4,$$
 则

$$\begin{vmatrix} 1 & a & 1 \\ 1 & 2 & a \\ a & b & 0 \end{vmatrix} = \underline{\hspace{2cm}}.$$

【答案】 8

【解析】



$$\begin{vmatrix} 1 & a & 1 \\ 1 & 2 & a \\ a & b & 0 \end{vmatrix} = 4 \neq 0, \quad r \begin{bmatrix} a & 0 & 1 \\ 1 & a & 1 \\ 1 & 2 & a \\ a & b & 0 \end{bmatrix} = 3, \quad r \begin{pmatrix} a & 0 & 1 & 1 \\ 1 & a & 1 & 0 \\ 1 & 2 & a & 0 \\ a & b & 0 & 2 \end{pmatrix} = 3, \quad \begin{vmatrix} a & 0 & 1 & 1 \\ 1 & a & 1 & 0 \\ 1 & 2 & a & 0 \\ a & b & 0 & 2 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 1 & a & 1 \\ 1 & 2 & a \\ a & b & 0 \end{vmatrix} - 2 \begin{vmatrix} a & 0 & 1 \\ 1 & a & 1 \\ 1 & 2 & a \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & a & 1 \\ 1 & 2 & a \\ a & b & 0 \end{vmatrix} = 8$$

16. 设随机变量  $X$  与  $Y$  相互独立, 且  $X \sim B(1, p), Y \sim B(2, p), p \in (0, 1)$ , 则  $X+Y$  与  $X-Y$  的相关系数为\_\_\_\_\_.

**【答案】**  $p(p-1)$

$$\text{Cov}(X+Y, X-Y) = DX - DY$$

**【解析】**

$$\begin{aligned} &= p(1-p) - 2p(1-p) \\ &= -p(1-p) = p(p-1) \end{aligned}$$

三、解答题

17. 已知可导函数  $y=y(x)$  满足  $ae^x + y^2 + y - \ln(1+x) \cos y + b = 0$ , 且  $y(0) = 0, y'(0) = 0$ .

(1) 求  $a, b$  的值;

(2) 判断  $x = 0$  是否为  $y(x)$  的极值点.

**【解析】**

(1) 将  $(0, 0)$  代入得  $a + b = 0$

将  $y'(0) = 0$  代入  $ae^x + 2yy' + y' - \frac{1}{1+x} \cos y + \ln(1+x)(\sin y)y' = 0$

得  $a + 0 - 1 = 0$ , 所以  $a = 1 \quad b = -1$

(2) 由  $e^x + 2yy' + y' - \frac{1}{1+x} \cos y + \ln(1+x) \sin y \cdot y' = 0$

两边对  $x$  求导, 得:



$$e^x + 2(y')^2 + 2yy'' + y'' + \frac{1}{(1+x)^2} \cos y + \frac{1}{1+x} \sin y + \frac{1}{1+x} \sin y \cdot y' + \ln(1+x) \left[ \cos y \cdot (y')^2 + \sin y y' \right] = 0$$

代入, 得  $1 + y''(0) + 1 = 0$ ,  $y''(0) = -2 < 0$ ,  $x = 0$  为极大值.

18. 已知平面区域  $D = \{(x, y) | 0 \leq y \leq \frac{1}{x\sqrt{1+x^2}}, x \geq 1\}$ .

(1) 求  $D$  的面积;

(2) 求  $D$  绕  $x$  轴旋转所成旋转体的体积.

**【解析】**

$$(1) \int_1^{+\infty} \frac{1}{x\sqrt{1+x^2}} dx \stackrel{x=\tan t}{=} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\tan t \cdot \sec t} \cdot \sec^2 t dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sec t}{\tan t} dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc t dt = \ln(\sqrt{2} + 1)$$

$$(2) \int_1^{+\infty} \pi \left( \frac{1}{x\sqrt{1+x^2}} \right)^2 dx = \int_1^{+\infty} \pi \frac{1}{x^2(1+x^2)} dx = \int_1^{+\infty} \pi \left( \frac{1}{x^2} - \frac{1}{1+x^2} \right) dx = \pi \left( 1 - \frac{\pi}{4} \right)$$

19. 已知平面区域  $D = \{(x, y) | (x-1)^2 + y^2 \leq 1\}$ . 计算二重积分  $\iint_D |\sqrt{x^2 + y^2} - 1| dx dy$ .

**【解析】**

$$D_1 = \{(x, y) | x^2 + y^2 \leq 1, (x-1)^2 + y^2 \leq 1\}$$

$$D_2 = \{(x, y) | x^2 + y^2 > 1, (x-1)^2 + y^2 \leq 1\}$$

$$\text{原式} = \iint_{D_1} (1 - \sqrt{x^2 + y^2}) dx dy + \iint_{D_2} (\sqrt{x^2 + y^2} - 1) dx dy$$

$$\text{其中} \iint_{D_1} (1 - \sqrt{x^2 + y^2}) dx dy = 2 \int_0^{\frac{\pi}{6}} d\theta \int_0^1 (1-r) r dr + 2 \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} d\theta \int_0^{2\cos\theta} (1-r) r dr = \frac{13}{18} \pi - \frac{\sqrt{3}}{2} - \frac{10}{9}$$

$$\begin{aligned} \iint_{D_2} (\sqrt{x^2+y^2}-1) dx dy &= \iint_D (\sqrt{x^2+y^2}-1) dx dy - \iint_{D_1} (\sqrt{x^2+y^2}-1) dx dy = \int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^{2\cos\theta} (r-1) r dr \\ &+ \iint_{D_1} (1-\sqrt{x^2+y^2}) dx dy \\ &= \frac{22}{9} - \frac{5}{18}\pi - \frac{\sqrt{3}}{2} \end{aligned}$$

所以原式 =  $\frac{4}{3} + \frac{4}{9}\pi - \sqrt{3}$ .

20. (12分) 设函数  $f(x)$  在  $[-a, a]$  上具有 2 阶连续导数, 证明:

(1) 若  $f(0)=0$ , 则存在  $\xi \in (-a, a)$ , 使得  $f''(\xi) = \frac{1}{a^2}[f(a) + f(-a)]$ ;

(2) 若  $f(x)$  在  $(-a, a)$  内取得极值, 则存在  $\eta \in (-a, a)$  使得

$$|f''(\eta)| \geq \frac{1}{2a^2} |f(a) - f(-a)|.$$

**【解析】**

$$(1) f(x) = f(0) + f'(0)x + \frac{1}{2}f''(\xi)x^2$$

$$f(x) = f'(0)x + \frac{1}{2}f''(\xi)x^2$$

$$f(a) = f'(0)a + \frac{1}{2}f''(\varepsilon_2)a^2.$$

$$f(-a) = f'(0)(-a) + \frac{1}{2}f''(\varepsilon_1)a^2$$

$$f(-a) + f(a) = \frac{1}{2}[f''(\varepsilon_1) + f''(\varepsilon_2)]a^2$$

由介值定理可知平均值  $\frac{1}{2}[f''(\varepsilon_1) + f''(\varepsilon_2)] = \frac{f(-a) + f(a)}{a^2} = f''(\xi)$

$\therefore$  即证

(2) 设  $f(x)$  在  $x=x_0$  处取得极值 即  $x_0 \in (-a, a), f'(x_0)=0$

$$\therefore f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(\xi)}{2}(x-x_0)^2$$

代入  $x=-a, x=a$

$$f(-a) = f(x_0) + \frac{f''(\eta_1)}{2}(a+x_0)^2 \quad (1)$$

$$f(a) = f(x_0) + \frac{f''(\eta_2)}{2}(a-x_0)^2 \quad (2)$$

(2) - (1) 得

$$f(a) - f(-a) = \frac{f''(\eta_2)}{2}(a-x_0)^2 - \frac{f''(\eta_1)}{2}(a+x_0)^2$$

$$|f(a) - f(-a)| = \left| \frac{f''(\eta_2)}{2}(a-x_0)^2 - \frac{f''(\eta_1)}{2}(a+x_0)^2 \right|$$

$$\leq \left| \frac{f''(\eta)}{2}(a-x_0)^2 \right| + \left| \frac{f''(\eta)}{2}(a+x_0)^2 \right|$$

$$\leq \left| \frac{f''(\eta)}{2} \right| [(a-x_0)^2 + (a+x_0)^2]$$

$$= \left( \frac{f''(\eta)}{2} \right) (2a^2 + 2x_0^2)$$

$$= |f''(\eta)| (a^2 + x_0^2)$$

$$\leq |f''(\eta)| \cdot 2a^2, \quad \text{其中 } f''(\eta) = \max \{ f''(\eta_1), f''(\eta_2) \}$$

$$\therefore |f''(\eta)| \geq \frac{1}{2a^2} |f(a) - f(-a)|$$

21. 设矩阵  $A$  满足对任意  $x_1, x_2, x_3$  均有  $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ 2x_1 - x_2 + x_3 \\ x_2 - x_3 \end{pmatrix}$ .

(1) 求  $A$ ;

(2) 求可逆矩阵  $P$  与对角矩阵  $\Lambda$ , 使得  $P^{-1}AP = \Lambda$ .

**【解析】**

(1) 由题可知,  $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\therefore A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

(2)  $|A - \lambda E| = -(2 + \lambda)(\lambda - 2)(\lambda + 1) = 0$

$\therefore A$  中  $\lambda_1 = 2, \lambda_2 = -1, \lambda_3 = -2$

$A$  中  $\lambda_1$  对应的特征向量  $\alpha_1 = (4, 3, 1)^T$ .

$A$  中  $\lambda_2$  对应的特征向量  $\alpha_2 = \left(-\frac{1}{2}, 0, 1\right)^T$

$A$  中  $\lambda_3$  对应的特征向量  $\alpha_3 = (0, -1, 1)^T$

$\therefore P = (\alpha_1, \alpha_2, \alpha_3)$

$$P^{-1}AP = \begin{pmatrix} 2 & & \\ & -1 & \\ & & -2 \end{pmatrix}$$

22. 设随机变量变量  $X$  的概率密度为  $f(x) = \frac{e^x}{(1+e^x)^2}, -\infty < x < +\infty$ , 令  $Y = e^x$ .

- (1) 求  $X$  的分布函数;
- (2) 求  $Y$  的概率密度;
- (3)  $Y$  的期望是否存在?

**【解析】**

(1)  $F(x) = \int_{-\infty}^x f(t) dt \quad (-\infty < x < +\infty)$

$$= \int_{-\infty}^x \frac{e^t}{(1+e^t)^2} dt$$

$$= \int_{-\infty}^x \frac{d(e^t+1)}{(1+e^t)^2}$$

$$= \frac{1}{1+e^t} \Big|_{-\infty}^x$$

$$= 1 - \frac{1}{1+e^x}$$

(2) 当  $y > 0$  时

$$f_Y(y) = f_X(\ln y) \cdot \left| \frac{1}{y} \right| = \frac{y}{(1+y)^2} \cdot \frac{1}{y} = \frac{1}{(1+y)^2}$$

$$\therefore f_Y(y) = \begin{cases} \frac{1}{(1+y)^2} & y > 0 \\ 0 & \text{其它} \end{cases}$$

(3)  $EY = \int_0^{+\infty} \frac{y}{(1+y)^2} dy$ ,  $\frac{y}{(1+y)^2} \sim \frac{1}{y}$ , 所以期望不存在.